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Inflationary Policy and Welfare with Limited Credit Markets

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This paper considers the costs and benefits of inflation using a stochastic version of Townsend's turnpike model in which agents of each type are allowed to remain at a trading post for multiple periods. Numerical results show that moderate rates of inflation can be welfare-improving, but only when private credit markets are extremely limited. More generally, the existence of private credit markets curtails the ability of inflationary policy to do both harm and good. In addition, the welfare consequences of inflation depend on how much information about the economy the government has access to when implementing its policies. *Journal of Economic Literature Classification Numbers:* D52, E31.

1. INTRODUCTION

Milton Friedman's (1969, p. 34) prescription, that the money stock be continuously contracted so as to produce "a rate of price deflation that makes the nominal rate of interest equal to zero," constitutes optimal policy in many popular economic models. Brock's (1975) money-in-the-utility-function model, Benhabib and Bull's (1983) transactions costs model, and Cooley and Hansen's (1989) cash-in-advance model all describe general equilibrium environments in which Friedman's rule applies. Each of these models contains impediments to trade that provide a role for government-issued money in facilitating exchange; each model captures what Friedman (1969, p. 3) calls the "transactions" demand for

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money. Deflation, brought about by a monetary–fiscal policy combination that levies identical lump-sum taxes on all agents to finance a steady withdrawal of money from circulation, allows agents to overcome the impediments to trade at no cost.[1] Thus, the Friedman rule supports a Pareto-optimal allocation.

Money also facilitates trade in Townsend's (1980) turnpike model of exchange. Unlike the representative agent models cited above, however, the turnpike model introduces heterogeneity into agents' endowment patterns. While Friedman-style deflation supports a Pareto-optimal allocation, implementing the Friedman rule requires the government to levy different taxes on different agents. As emphasized by Sargent (1987, p. 224), therefore, the turnpike model serves to point out a potential difficulty with the Friedman rule: the government may need enough information about the economy and its inhabitants to distinguish between agents of different types.

Levine (1991) and Kehoe *et al.* (1992) take Townsend's work a step further by giving agents random endowments. Their stochastic version of the turnpike model continues to capture the transactions demand for money, in the sense that market frictions render nonmonetary trade impossible. But agents use money primarily to insure against negative income shocks, so this model accounts for a second motive that Friedman calls (1969, p. 3) the "asset" demand for money. In this model, an agent who experiences a streak of bad luck may run out of money in finite time. Thereafter, he cannot purchase goods as long as his hard luck continues. Equal lump-sum transfers of money from the government to all agents will, in such circumstances, permit otherwise infeasible transactions to occur.[2] Inflationary policy causes agents to economize on their holdings of real balances and thereby inhibits trade; this is the same cost of inflation that is identified by models of the transactions demand for money. At the same time, however, inflationary policy prevents agents from running out of cash when they need it most; this is a benefit of inflation that is associated with the asset demand for money. The benefits often exceed the costs when the government cannot distinguish between agents of

[1] Wallace (1980) identifies monetary policy with asset exchange and fiscal policy with asset creation. In his terms, the monetary side of this deflationary policy involves open market exchanges of government bonds for money, while the fiscal side involves retiring the bonds using the proceeds from lump-sum taxation. All of the models cited here combine the monetary and fiscal authority into a single government entity, so that deflation can be described as resulting directly from a monetary contraction financed by lump-sum taxation. It is clear, nonetheless, that elements of both monetary and fiscal policy are required to implement the Friedman rule.

[2] Note that like the deflationary policy called for by the Friedman rule, the inflationary policy described here involves elements of both monetary and fiscal policy.

different types, so that moderate inflation is preferable to both zero inflation and Friedman-style deflation.

This paper follows Levine (1991) and Kehoe *et al.* (1992) by considering the costs and benefits of inflationary policy in a stochastic version of Townsend's turnpike model. But in addition to extending Townsend's (1980) original specification by randomly distributing the aggregate endowment between two sets of agents, this paper also allows agents to remain at a trading post for multiple periods, as first suggested by Manuelli and Sargent (1992). This additional extension makes the turnpike model presented here more general than previous versions along one key dimension. In the earlier models, noninterest-bearing, government-issued money is the only financial asset; here, agents have access to markets for private securities as well.

Assumptions that rule out the existence of private credit markets are common, not only in previous work with stochastic turnpike models, but in all work that examines the asset demand for money. Bewley (1980, 1983), Scheinkman and Weiss (1986), and Imrohoroglu (1992), for example, all present models in which money is the sole asset with which agents can insure against income fluctuations. Friedman himself notes (1969, p. 3), however, that in reality there are assets besides money that agents accumulate for insurance purposes. Allowing for longer trading periods in the turnpike model, so that money and private securities coexist, therefore recognizes an aspect of reality that related studies often ignore.

The paper focuses on the two potential difficulties with the Friedman rule that are identified in earlier work with the turnpike model. The first set of results is obtained, following Levine (1991) and Kehoe *et al.* (1992), under the assumption that the government cannot respond to the state of the economy or distinguish between agents of different types when implementing its policies. The results demonstrate that the beneficial effects of inflation found in earlier work are quantitatively significant only when the length of the trading session is such that money is the only asset. When the trading session is extended so that money and private securities coexist, inflation ceases to be welfare-improving. The results suggest, therefore, that the insurance-providing role of inflationary policy, brought about by the asset demand for money, is likely to be important only where credit markets are extremely attenuated. Where even limited credit markets exist, the dominant effect of inflation is to inhibit trade, exactly as in models of the transactions demand for money. More generally, the results show that the power of inflationary policy to do both harm and good diminishes as the scope for private credit arrangements expands. Assumptions that shut down private credit markets in models of the asset demand for money are, therefore, important determinants of these models' policy implications.

The second set of results returns to the informational issues first raised by Townsend (1980). As in Townsend's original model, the Friedman rule supports a Pareto-optimal allocation in the stochastic turnpike model, but requires that the government possess enough information to discriminate between agents of different types. Even when the government cannot distinguish between agent types, however, policies exist that feature zero inflation on average and are welfare-improving relative to those that generate consistently positive inflation rates. These policies are feasible only when some of the informational constraints imposed on the government by Levine (1991) and Kehoe *et al.* (1992) are relaxed, but require less information to implement than the Friedman rule. The results echo Sargent's (1987, p. 270) conclusion that "whether or not inflation is 'bad' ··· depends on how new currency is injected into the system." The results show, therefore, that informational assumptions that constrain the ways in which the government can increase the money supply are also important in determining the policy implications of models of the asset demand for money.

In describing an environment where agents have an asset demand for money and where money and private securities coexist, this paper is related to several others in the recent literature. Manuelli (1990) presents a stochastic version of the overlapping generations (OG) model to show that money can have value because it serves, in part, as a vehicle for insurance. Like the turnpike model presented here, Manuelli's OG model provides an example of how the asset demand for money arises when some markets for private securities are closed. Aiyagari (1988) takes the deterministic OG model and extends the lifetime of each generation beyond the usual two periods. He thereby obtains a model in which some transactions are accomplished through the exchange of private securities while others require government-issued money. Aiyagari finds that the role for money diminishes as the length of each generation's lifetime extends to infinity, just as it does in the stochastic turnpike model as the length of each trading session grows. Finally, Azariadis and Smith (1993) develop a stochastic OG model with private information that illustrates how the introduction of money can actually curtail insurance possibilities for one class of agents; in the stochastic turnpike model, in contrast, money expands insurance possibilities for all agents. None of these three papers conducts a quantitative analysis of welfare like the one presented here. But by demonstrating that the OG and turnpike models can differ along other dimensions, Azariadis and Smith's work suggests that such an analysis would likely yield additional insights.[3]

[3] Townsend (1980) shows that the welfare properties of monetary equilibria in his deterministic turnpike model differ from those in the deterministic OG model. His results also suggest that a quantitative investigation of welfare in stochastic monetary OG models would yield results that differ from those obtained here with the stochastic turnpike model.

The stochastic turnpike model is presented in the next section. A competitive equilibrium for the economy is defined in Section 3 and characterized in Section 4. Section 5 then assesses the costs and benefits of inflationary policy in the turnpike model, focusing on the question of whether the results of Levine (1991) and Kehoe *et al.* (1992) continue to apply when the scope for private credit arrangements expands. Section 6 describes how policy is constrained by the content of the government's information set and shows how the effects of inflation depend crucially on how policy can be implemented. Section 7 briefly reviews the results and suggests directions for future research.

2. THE STOCHASTIC TURNPIKE MODEL

As in Townsend (1980) and Manuelli and Sargent (1992), the economy consists of a large number of infinite-lived agents who are distributed uniformly into a countable number of trading posts located at discrete intervals along a turnpike of infinite length. The trading posts are spatially isolated, meaning that agents at one post can neither trade nor communicate with agents at other posts.

The agents are of two types, indexed by $i \in \{0, 1\}$; at any given time $t = 1, 2, 3, \ldots$, there are equal numbers of each type at each trading post. At the end of each period $t = N, 2N, 3N, \ldots$, where $N \geq 1$, each agent of type 0 leaves his trading post, arriving at the next post to the east along the turnpike at the beginning of the following period. Simultaneously, each type 1 agent moves one market to the west. Thus, agents of different types remain together for a single N-period trading session before moving on, never to meet again.

Uncertainty is introduced into the turnpike model by randomly distributing the aggregate endowment between two sets of agents. In each period $t = 1, 2, 3, \ldots$, the endowment shock $s_t \in \{0, 1\}$ is revealed to the agents. The timing of this revelation is such that when an agent is moving between two posts he leaves the first before, but arrives at the second after, s_t becomes known. If $s_t = 0$, then each type 0 agent in the economy receives an endowment of ω_H units of a perishable consumption good in period t, while each type 1 agent receives only $\omega_L < \omega_H$ units of the good. If $s_t = 1$, then type 1 agents get the high endowment ω_H in period t, while type 0 agents get the low endowment ω_L. The constant size of the aggregate endowment is normalized so that $\omega_H + \omega_L = 1$. The shock s_t is independently and identically distributed (iid) over time, with equal probability of either outcome.

Agents have common preferences over state-contingent consumption, represented by the additively time-separable expected utility function

$$E\left\{\sum_{t=1}^{\infty} \beta^t u(c_t)\right\},$$

where $\beta \in (0, 1)$ is the discount factor, c_t is consumption at date t, and $u(\cdot)$ is strictly increasing and strictly concave.

Preference and endowment patterns are such that trade will only take place intertemporally between agents of different types. Itineraries are such that private debt contracts can be used to facilitate trade between agents only across periods during which they remain paired at the same trading post. As the only available outside asset, government-issued money is valued in this environment since, unlike private securities that must be redeemed before agents move on, cash can be carried across trading posts.[4] The importance of money in the economy depends inversely on the length N of each trading session. With $N = 1$, private credit markets are ruled out completely; as in the earlier models of Bewley (1980, 1983), Scheinkman and Weiss (1986), Levine (1991), Kehoe *et al.* (1992), and Imrohoroglu (1992), money is the only financial asset. As N increases, the scope for private credit markets expands and the role for money diminishes.

The government supplies each agent of type i with M_0^i units of money prior to the beginning of period 1, where a normalization yields $M_0^0 + M_0^1 = 1$. It augments (or contracts) this initial supply with lump-sum transfers (or taxes) of amount H_t^i to each agent of type i at the beginning of each period $t = 1, 2, 3, \ldots$. Thus, with twice the per-capita money supply at the end of period t denoted by M_t^s and with $M_0^s = 1$,

$$M_t^s = H_t^0 + H_t^1 + M_{t-1}^s$$

for all $t = 1, 2, 3, \ldots$. In general, the transfers at time t will be allowed to depend on the realization of $s^t = \{s_1, s_2, \ldots, s_t\}$, the history of endowment shocks through period t, so that $H_t^i = H^i(s^t)$ and $M_t^s = M^s(s^t)$.

3. COMPETITIVE EQUILIBRIUM DEFINED

Following Townsend (1980) and Manuelli and Sargent (1992), attention is confined here to symmetric equilibria, meaning those in which agents with identical endowments and money holdings receive identical consumption allocations. Also, just as the earlier studies focus on equilibria with periodic outcomes, reflecting periodic endowment specifications, the analysis here focuses on equilibria in which outcomes are described

[4] Equilibria no doubt exist in which private debt contracts do get carried across trading posts, have positive value, and are never redeemed by their issuers. Townsend (1980, p. 269) notes, however, that such inconvertible, privately-issued exchange media would be "virtually indistinguishable" from government-issued fiat money. Hence it is assumed here that money is the only financial asset that is carried across trading posts in equilibrium.

by time-invariant functions of a minimal number of state variables, reflecting the simple structure of the endowment process.[5] Under these additional assumptions, competitive equilibria for the stochastic turnpike economy may be completely characterized by describing the optimizing behavior of two representative agents (one of each type) and the market clearing process at a single representative trading post.

The restriction that private debt contracts cannot be extended beyond an N-period trading session, taken together with the assumptions of the time-separability of preferences, the iid structure of shocks, and the absence of storage possibilities, imply that the infinite horizon economy may be studied as a sequence of N-period economies. Each member of the sequence is linked to its predecessor and follower by the distribution of cash balances across agent types at the beginning and end of each trading session. At each date $t \in \mathbf{T} = \{1, N + 1, 2N + 1, \ldots\}$, agents trade claims for state-contingent delivery of consumption goods and money over the following $N - 1$ periods, taking the realized history $s^t = \{s_1, s_2, \ldots, s_t\}$ as given, since s^t is revealed before agents arrive at their new posts at the beginning of time t.

Thus, for any $t = 1, 2, 3, \ldots$, let $y^i(s^t)$ and $c^i(s^t)$ denote the endowment and consumption of a representative type i agent in period t after the history s^t has been realized. Similarly, let $m^i(s^{t-1})$ denote the fraction of the total money supply held at the end of period $t - 1$, and hence carried into period t, by the representative type i agent after the history s^{t-1} has been realized. Since the endowment process is iid, $y^i(s^t) = y^i(s_t)$, with $y^0(0) = y^1(1) = \omega_H$ and $y^0(1) = y^1(0) = \omega_L$.

For any $t \in \mathbf{T}$, let $\mathbf{h}^t$ denote the set of all possible histories s^t through time t. For any $s^t \in \mathbf{h}^t$ and any $j = 0, 1, \ldots, N - 1$, let $\mathbf{h}^j(s^t)$ denote the set of all histories $s^{t+j} = \{s_1, \ldots, s_t, \ldots, s_{t+j}\}$ that can possibly follow s^t. For any $s^{t+j} \in \mathbf{h}^j(s^t)$, let $q_t(s^{t+j})$ denote the price, in time t following the realization of s^t, of a claim to one unit of consumption in time $t + j$ upon the realization of s^{t+j}, in terms of time t consumption. Similarly, let $p_t(s^{t+j})$ denote the price, in time t following the realization of s^t, of a claim to one unit of money in time $t + j$ upon the realization of s^{t+j}, also in terms of time t consumption. Note that $q_t(s^t) = 1$ by definition, while $p_t(s^t)$ is the inverse of the price level at time t.

As sources of funds during each trading session, the representative type i agent has the money that he carries into the session, the state-contingent endowment stream that he receives during the session, and the state-contingent government transfers that he receives during the session. A neutrality argument such as those used throughout Sargent (1987) indicates that for each $t \in \mathbf{T}$ and $s^t \in \mathbf{h}^t$, only the present discounted value of

[5] This is not to say that an investigation of nonsymmetric or nonstationary equilibria would not be of interest, only that it is left here as a task for future research.

the lump-sum government transfers received over the entire course of the time t trading session will enter into the representative type i agent's decision-making process; the timing of these transfers does not matter. It can therefore be assumed without loss of generality that all transfers are made in the initial period of each trading session, so that $H^i(s^{t+j}) = 0$ for all $s^{t+j} \in \mathbf{h}^j(s^t)$ with $j \neq 0$.

As uses of funds in each trading session, the representative type i agent has his state-contingent consumption stream as well as the money that he wishes to carry into the next trading session. The spatial organization of exchange implies that the agent's sources and uses of funds must balance over the course of each N-period trading session. Thus, the representative type i agent's problem can be stated formally as:

Problem. Given M_0^i and given $y^i(s^{t+j})$, $H^i(s^t)$, $q_t(s^{t+j})$, and $p_t(s^{t+j})$ for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N-1$, and $s^{t+j} \in \mathbf{h}^j(s^t)$, choose $c^i(s^{t+j})$ and $m^i(s^{t+N-1})$ for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N-1$, $s^{t+j} \in \mathbf{h}^j(s^t)$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$ to maximize

$$\sum_{t\in\mathbf{T}} \sum_{s^t\in\mathbf{h}^t} \sum_{j=0}^{N-1} \sum_{s^{t+j}\in\mathbf{h}^j(s^t)} (\beta/2)^{t+j} u[c^i(s^{t+j})]$$

subject to

$$\begin{aligned} p_t(s^t)[m^i(s^{t-1})M^s(s^{t-1}) &+ H^i(s^t)] \\ &+ \sum_{j=0} \sum_{s^{t+j}\in\mathbf{h}^j(s^t)} q_t(s^{t+j}) y^i(s^{t+j}) \geq \sum_{j=0}^{N-1} \sum_{s^{t+j}\in\mathbf{h}^j(s^t)} q_t(s^{t+j}) c^i(s^{t+j}) \\ &+ \sum_{s^{t+N-1}\in\mathbf{h}^{N-1}(s^t)} p_t(s^{t+N-1}) m^i(s^{t+N-1}) M^s(s^{t+N-1}) \end{aligned}$$

for all $t \in \mathbf{T}$ and $s^t \in \mathbf{h}^t$ and

$$1 \geq m^i(s^{t+N-1}) \geq 0$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$.

The objective function in this problem is the representative type i agent's expected utility, written to reflect the fact that the probability that any history s^{t+j} will be realized through time $t + j$ is just $(1/2)^{t+j}$. The first constraint is the agent's budget constraint, which formalizes the requirement that his sources and uses of funds must balance over the course of each trading session. Thus, the money that he carries into the trading session, his government transfer, and his state-contingent endowment stream appear on the left-hand side; his state-contingent consumption stream and the money that he carries out of the trading session appear on the right. The second constraint simply indicates that the agent's share of the money supply must be a fraction between zero and one.

A competitive equilibrium for the stochastic turnpike economy is defined as a set of prices and quantities such that all agents are optimizing and all markets clear:

DEFINITION. A *competitive equilibrium* consists of the initial conditions M_0^0 and M_0^1 and the prices and quantities $y^i(s^{t+j})$, $H^i(s^t)$, $c^i(s^{t+j})$, $m^i(s^{t+N-1})$, $q_t(s^{t+j})$, and $p_t(s^{t+j})$ for $i \in \{0, 1\}$, all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N-1$, $s^{t+j} \in \mathbf{h}^j(s^t)$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$ such that

(i) For $i \in \{0, 1\}$, the $c^i(s^{t+j})$ and $m^i(s^{t+N-1})$ solve the representative type i agent's problem given the M_0^i, $y^i(s^{t+j})$, $H^i(s^t)$, $q_t(s^{t+j})$, and $p_t(s^{t+j})$.

(ii) The $y^i(s^{t+j})$, $c^i(s^{t+j})$, and $m^i(s^{t+N-1})$ satisfy

(a) $c^0(s^{t+j}) + c^1(s^{t+j}) = y^0(s^{t+j}) = y^1(s^{t+j}) = 1$

(b) $m^0(s^{t+N-1}) + m^1(s^{t+N-1}) = 1$.

4. COMPETITIVE EQUILIBRIUM CHARACTERIZED

The representative type i agent's first order conditions are

$$(\beta/2)^j u'[c^i(s^{t+j})] = q_t(s^{t+j})u'[c^i(s^t)] \tag{1}$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N-1$, and $s^{t+j} \in \mathbf{h}^j(s^t)$ and

$$p_t(s^{t+N-1})u'[c^i(s^t)] \geq E_{t+N-1}\{(\beta^N/2^{N-1})p_{t+N}(s^{t+N})u'[c^i(s^{t+N})]\}, \tag{2}$$

with equality if $m^i(s^{t+N-1}) > 0$, for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$, where $E_{t+N-1}\{\cdot\}$ indicates that the expectation is conditional on time $t + N - 1$ information, that is, conditional on the realization of s^{t+N-1}.

Equation (1), along with the market clearing condition for goods listed in the definition of competitive equilibrium, implies that $q_t(s^{t+j}) = (\beta/2)^j$ and that within each trading session, each agent's consumption is constant across date–state combinations; that is,

$$c^i(s^{t+j}) = c^i(s^t) \tag{3}$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N-1$, and $s^{t+j} \in \mathbf{h}^j(s^t)$. Equation (3) indicates that marginal utilities are equated and hence the gains from trade are exhausted within each trading session, when private credit markets are operative. By trading in private securities at date $t \in \mathbf{T}$, agents in the stochastic turnpike model are able to fully insure against the shocks $\{s_{t+1}, s_{t+2}, \ldots, s_{t+N-1}\}$ that are realized during the time t trading session.

The absence of arbitrage opportunities in the money market guarantees that

$$p_t(s^t) = \sum_{s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)} p_t(s^{t+N-1}) \tag{4}$$

will hold for all $t \in \mathbf{T}$ and $s^t \in \mathbf{h}^t$. Otherwise, any agent could buy (sell) nominal balances at the t after s^t has been realized, simultaneously sell (buy) claims to equal quantities of nominal balances for every history $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$, and thereby make a certain profit. Summing (2) over all $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$ and using (4) yields a stochastic Euler equation of the form

$$p_t(s^t)u'[c^i(s^t)] \geq E_t\{\beta^N p_{t+N}(s^{t+N})u'[c^i(s^{t+N})]\}, \tag{5}$$

which holds with equality when $m^i(s^{t+N-1}) > 0$ for all $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$.

The nonnegativity constraint on money holdings makes the intersession Euler equation (5) an inequality. As emphasized by Zeldes (1989), even if the nonnegativity constraint does not bind in the current period, the possibility that it will bind in some future period is enough to lower consumption today. Thus, even when it holds with equality, Eq. (5) indicates that the gains from trade are only partially exhausted across trading sessions, when private credit markets fail. As the only available outside asset, government-issued money is a valuable, but generally imperfect, vehicle for agents to insure against the shock s_{t+N} that is realized between trading sessions.[6]

Note that for $t \in \mathbf{T}$, the distribution of cash balances held at the end of time $t - 1$, and hence carried into time t, following the realization of history s^{t-1} can be summarized by the number

$$m_{t-1} = m^0(s^{t-1}) = 1 - m^1(s^{t-1}).$$

Note also that government policy can be described completely by the initial conditions M_0^0 and M_0^1, the gross rate of money supply growth per trading session,

[6] In other words, money is an imperfect substitute for the private securities that cannot be carried across trading sessions. Within trading sessions, on the other hand, a standard no-arbitrage argument indicates that deflation must prevail, so that money and private securities are perfect substitutes.

$$z(s^t) = [H^0(s^t) + H^1(s^t) + M^s(s^{t-1})]/M^s(s^{t-1}) = M^s(s^{t+N-1})/M^s(s^{t-1}), \quad (6)$$

and the size of the nominal transfer to the representative type 0 agent as a fraction of the total money supply,

$$h(s^t) = H^0(s^t)/M^s(s^{t-1}) = [M^s(s^{t+N-1}) - M^s(s^{t-1}) - H^1(s^t)]/M^s(s^{t-1}). \quad (7)$$

Since agents are able to fully insure against all within-session shocks, but cannot carry private securities across trading posts, m_{t-1} and s_t are the only variables that distinguish the time t trading session from any other. The symmetry and stationarity assumptions made in the previous section imply, therefore, that equilibrium prices and quantities can be expressed as time-invariant functions of m_{t-1} and s_t. In particular, consumption and money holdings are described by the functions $c(m, s)$ and $\mu(m, s)$, where

$$c^0(s^t) = 1 - c^1(s^t) = c(m_{t-1}, s_t)$$

and

$$m^0(s^{t+N-1}) = 1 - m^1(s^{t+N-1}) = \mu(m_{t-1}, s_t),$$

while government policy is described by the functions $z(m, s)$ and $h(m, s)$, where

$$z(s^t) = z(m_{t-1}, s_t)$$

and

$$h(s^t) = h(m_{t-1}, s_t).$$

Since

$$m_{t+N-1} = \mu(m_{t-1}, s_t),$$

the equilibrium function $\mu(m, s)$ governs how the distribution of money balances across agent types evolves over time. For $t \in \mathbf{T}$, let $\lambda_t(m)$ denote the probability, prior to $t = 1$, that type 0 agents will carry the fraction m of the total money supply into the time t trading session. In each of the examples considered below, the function $\mu(m, s)$ is such that $\lambda_t(m)$ converges to a limiting distribution as t approaches infinity, as in Foley and Hellwig (1975), regardless of the initial conditions M_0^0 and M_0^1. In each case, therefore, the economy reaches a stochastic steady state in which prices and quantities are strictly stationary random variables.

Because the stochastic turnpike model features heterogeneous agents, it is necessary to specify precisely which welfare criterion is to be used in evaluating policy experiments. In each of the examples considered below, it is assumed, following Taub (1988), that the initial conditions M_0^0 and M_0^1 and the functions $z(m, s)$ and $h(m, s)$ are determined simultaneously by starting the economy up in its stochastic steady state. Prior to the initial period, the government announces its choices for z and h. It then chooses $M_0^0 = 1 - M_0^1$ randomly, giving the representative type 0 agent a fraction m of the initial money supply with probability $\lambda(m)$, where λ is the limiting distribution of money balances under the policy described by z and h. This strategy amounts to choosing z and h before agents have been distinguished by type and therefore allows each policy to be evaluated according to the ex-ante expected utility that it provides to any representative agent.[7]

5. THE COSTS AND BENEFITS OF INFLATIONARY POLICY

Since agents hold money primarily to insure against the endowment shocks that are realized between trading sessions, the stochastic turnpike model captures the asset demand for money. Levine (1991) and Kehoe *et al.* (1992) discover that in their model of the asset demand for money, inflationary policies have insurance-providing benefits that often make a positive rate of inflation preferable to zero inflation. In this earlier model, however, money is the only financial asset. Using the stochastic turnpike model, this section applies numerical methods to determine whether inflation continues to be welfare-improving when money and private credit instruments coexist.

To perform the numerical work, the utility function is specialized to the constant elasticity of substitution (CES) form,

$$u(c) = (c^{1-\sigma} - 1)/(1 - \sigma), \qquad \sigma > 0.$$

In order to focus on the effects of relaxing the constraints that limit the existence of private credit markets, the preference parameters are held fixed throughout most of the analysis, with $\beta = 0.95$ and $\sigma = 2$; to investigate the robustness of the results, however, experiments with log utility ($\sigma = 1$) are also discussed.[8] The high and low endowments are also held fixed; with $\omega_H = 0.99$ and $\omega_L = 0.01$, agents have strong incentives

[7] Levine (1991) and Kehoe *et al.* (1992) adopt a similar ex-ante welfare criterion.

[8] The brief review of the literature in Mehra and Prescott (1985) cites estimates of σ that are generally between 1 and 2; the settings for σ used here bracket this range.

for trade but autarky is still possible. The appendix outlines the numerical solution procedure.

Levine (1991) and Kehoe *et al.* (1992) constrain government policy so that it cannot respond to the state of the economy or distinguish between agents of different types. In the context of the stochastic turnpike model presented here, this class of policies corresponds to those that call for a constant growth rate of the money supply effected by equal lump-sum transfers to all agents. With $H^0(s^t) = H^1(s^t)$, Eqs. (6) and (7) imply that the functions $z(m, s)$ and $h(m, s)$ must satisfy

$$z(m, s) = z \tag{8}$$

and

$$h(m, s) = (z - 1)/2 \tag{9}$$

for all $m \in [0, 1]$ and $s \in \{0, 1\}$. Note that in the stochastic turnpike model, a steady rate of monetary expansion does not generally give rise to a constant inflation rate. Since the aggregate endowment is fixed at unity and since $p_t(s^t)$ is the inverse of the price level at time t, the equation of exchange $MV = PY$ indicates that the income velocity of money in this economy is described by the function $v(m, s)$, where

$$v(m_{t-1}, s_t) = v(s^t) = 1/[p_t(s^t)M^s(s^{t-1})].$$

Velocity varies over time with the state (m_{t-1}, s_t) of the economy, so the inflation rate equals the money growth rate only on average.

When the functions $z(m, s)$ and $h(m, s)$ are constrained to satisfy (8) and (9), the stochastic turnpike model becomes quite similar to the model studied by Levine (1991) and Kehoe *et al.* (1992). Private credit markets are shut down altogether, money is the only asset, and the government cannot distinguish between agent types. In fact, the benefits of inflationary policy found in the earlier model are also present here.

Table I summarizes the effects of various constant money growth rates in the turnpike economy with $N = 1$.[9] Since z is the growth rate of money per trading session, $z^{1/N}$ is the growth rate of money per period. At first, expected utility in the stochastic steady state increases with the rate of money growth. Welfare is highest when the money supply expands by 3% per period. For rates above 3% agents are made progressively worse off by additional money growth.

[9] Only nonnegative money growth rates are considered, since work by Bewley (1980, 1983) indicates that monetary equilibria may fail to exist under deflation when agents have an asset demand for money and policy must satisfy constraints like (8) and (9).

TABLE I
CONSTANT MONETARY GROWTH: $\sigma = 2$, $N = 1$

Money growth per period $z^{1/N}$	Variance of consumption $V[c(m, s)]$	Average velocity $E[v(m, s)]$	Expected utility	Welfare cost ω^*
1.00	0.0142	0.2578	−23.1909	1.0000
1.01	0.0148	0.3108	−23.1784	0.9997
1.02	0.0156	0.3473	−22.9980	0.9954
1.03	0.0163	0.3820	−22.9685	0.9947
1.04	0.0170	0.4171	−22.9861	0.9951
1.05	0.0177	0.4511	−23.0991	0.9978
1.06	0.0184	0.4844	−23.2287	1.0009
1.07	0.0192	0.5173	−23.3708	1.0043
1.08	0.0199	0.5502	−23.5063	1.0075
1.09	0.0205	0.5828	−23.6096	1.0099
1.10	0.0212	0.6150	−23.7524	1.0133
1.12	0.0225	0.6797	−24.0380	1.0201
∞	0.2401	∞	−940.5960	22.7411

Table I reveals that agents economize on their real cash balances in response to higher rates of inflation, so that average velocity increases with z. As a result, the volume of trade falls and the variance of each agent's consumption rises with z. On the other hand, the transfers associated with a positive money growth rate ensure that agents never run out of cash and guarantee that trade will be feasible precisely when it is needed most. Here, as in Kehoe *et al.* (1992), there is a trade-off between these two effects of inflation. The positive effect dominates at low rates of money growth, making moderate inflation desirable.

The benefits of inflation are greatly reduced, however, when agents have limited access to private credit markets. For the turnpike model with $N = 2$, where agents can issue private securities every other period, Table II shows that welfare peaks when money growth is less than 0.1% per period. For $N = 3$ and $N = 4$, where private securities play a larger role, Tables III and IV indicate that the harmful effects of inflation dominate even at very low rates of money growth.

Kehoe *et al.* (1992) find that monetary expansion is most likely to be desirable when endowments are highly variable. If the ratio ω_L/ω_H is small, then an agent who is short on money balances and receives the low endowment has a much higher marginal utility of consumption than an agent who has a large cash hoard and receives the high endowment. The extra money received through government transfers is then extremely valuable in facilitating trade. When credit markets are open, however, an unlucky agent with no money can still obtain goods by issuing private

TABLE II
CONSTANT MONETARY GROWTH: $\sigma = 2$, $N = 2$

Money growth per period $z^{1/N}$	Variance of consumption $V[c(m, s)]$	Average velocity $E[v(m, s)]$	Expected utility	Welfare cost ω^*
1.000000	0.012178	0.674153	−21.219474	1.000000
1.000001	0.012177	0.674163	−21.219449	0.999999
1.00001	0.012177	0.674250	−21.219221	0.999994
1.0001	0.012167	0.675148	−21.216643	0.999930
1.001	0.012235	0.683414	−21.228003	1.000212
1.01	0.0126	0.7735	−21.2768	1.0014
1.02	0.0138	0.8855	−21.4751	1.0064
1.04	0.0159	1.1403	−21.8474	1.0156
1.06	0.0172	1.3620	−22.0759	1.0213
1.08	0.0185	1.5823	−22.2863	1.0265
1.10	0.0196	1.8092	−22.4728	1.0312
1.12	0.0209	2.0535	−22.6974	1.0367
∞	0.0631	∞	−31.8409	1.2641

securities. Thus, the transfers in the turnpike model become less important as N increases.

Table I–IV also report the welfare costs of the constant money growth rate policies. For each value of N, the welfare costs are calculated by finding the quantity ω^* such that the expected utility of a representative agent in the economy with $z > 1$, a high endowment of $\omega^*\omega_H$, and a low endowment of $\omega^*\omega_L$ is equal to the expected utility of a representative

TABLE III
CONSTANT MONETARY GROWTH: $\sigma = 2$, $N = 3$

Money growth per period $z^{1/N}$	Variance of consumption $V[c(m, s)]$	Average velocity $E[v(m, s)]$	Expected utility	Welfare cost ω^*
1.000000	0.010407	1.309977	−20.741075	1.000000
1.000001	0.010407	1.310008	−20.741079	1.000000
1.00001	0.010407	1.310286	−20.741118	1.000001
1.0001	0.010410	1.313137	−20.741542	1.000012
1.001	0.010459	1.336838	−20.748719	1.000192
1.01	0.0112	1.5619	−20.8617	1.0030
1.02	0.0118	1.8123	−20.9668	1.0057
1.04	0.0136	2.4709	−21.2605	1.0131
1.06	0.0154	3.3488	−21.5671	1.0208
1.08	0.0159	3.9285	−21.6496	1.0229
1.10	0.0167	4.6330	−21.7668	1.0258
1.12	0.0176	5.5034	−21.9122	1.0295
∞	0.0295	∞	−24.0855	1.0842

TABLE IV
CONSTANT MONETARY GROWTH: $\sigma = 2$, $N = 4$

Money growth per period $z^{1/N}$	Variance of consumption $V[c(m, s)]$	Average velocity $E[v(m, s)]$	Expected utility	Welfare cost ω^*
1.000000	0.008931	2.342301	−20.440133	1.000000
1.000001	0.008931	2.342375	−20.440140	1.000000
1.00001	0.008931	2.343039	−20.440203	1.000002
1.0001	0.008935	2.349696	−20.440834	1.000018
1.001	0.009037	2.416420	−20.457555	1.000442
1.01	0.0100	3.1670	−20.6124	1.0044
1.02	0.0103	3.6523	−20.6524	1.0054
1.04	0.0111	4.9786	−20.7831	1.0087
1.06	0.0122	7.1386	−20.9676	1.0134
1.08	0.0136	11.2163	−21.1951	1.0191
1.10	0.0152	21.5922	−21.4582	1.0258
1.12	0.0169	98.2424	−21.7521	1.0333
∞	0.0174	∞	−21.8506	1.0358

agent in the economy with $z = 1$, a high endowment of ω_H, and a low endowment of ω_L. Thus, ω^* measures the increase in average endowment that compensates a representative agent for a positive rate of money growth and, hence, a positive average inflation rate. As noted by Imrohoroglu (1992), this welfare cost measure explicitly accounts for the increase in consumption variability caused by higher inflation.

Table I indicates that for $N = 1$, the net benefit of a 3% average inflation rate relative to a zero average inflation rate is equal to slightly more than 0.5% of average income. The welfare cost of a 10% average inflation rate relative to zero inflation is 1.33% of average income. Comparing the costs of inflation when $N \geq 2$ to those with $N = 1$ reveals two ways in which these costs change as agents gain access to private credit markets. Because these two effects work in opposite directions, one cannot say that the cost of any given rate of inflation increases or decreases unambiguously as private securities replace money in increasing numbers of trades.

As discussed above, the positive effects of monetary expansion diminish in importance as N increases. Through this effect, the welfare cost of any given inflation rate tends to increase with the length of each trading session. However, since money is used in fewer transactions, the reduction in monetary trade resulting from inflation becomes less significant as N increases. Through this effect, the cost of inflation tends to decrease with the length of each trading session. Tables I–IV show that the first of these effects tends to dominate at low rates of money growth, so that the costs of moderate inflations increase with N. Eventually the second effect takes over, so that the costs of more rapid inflations decrease with N. Thus, the cost of an 8% inflation first rises from 0.75% of income when $N = 1$ to 2.65% of income when $N = 2$, then falls to 2.29% of income with $N = 3$ and 1.91% of income when $N = 4$. Even this pattern is not completely uniform, however. Although the cost of a 10% inflation rises from 1.33% of income with $N = 1$ to 3.12% of income with $N = 2$, then falls to 2.58% of income with $N = 3$, it remains at 2.58% of income with $N = 4$. Similarly, the cost of a 12% inflation rises from $N = 1$ to $N = 2$, falls from $N = 2$ to $N = 3$, then rises again from $N = 3$ to $N = 4$.

The tables illustrate that the government's ability to improve welfare through inflationary policy becomes extremely limited as the trading session is lengthened. At the same time, however, its ability to damage the economy is also curtailed as N increases. Bailey (1956) notes that the welfare cost of hyperinflationary rates of monetary expansion can be bounded from above by computing the cost to the economy of abandoning money altogether. Tables I–IV report this upper bound for each turnpike economy in the rows labeled "$z^{1/N} = \infty$." Autarky is the only alternative to monetary trade in the model with $N = 1$. The average endowment must be increased by over 2200% to make agents as well off under autarky as they are in a monetary equilibrium with a fixed money supply; thus, the costs of hyperinflation when $N = 1$ can be quite high. For $N \geq 2$, considerable amounts of trade can take place without the use of money, so that the upper bound on the welfare cost of hyperinflation is 26.4% of average income when $N = 2$, 8.4% of income when $N = 3$, and only 3.6% of income when $N = 4$.

Previous estimates of the welfare cost of a sustained 10% inflation include the figure of 0.4% of income found by Cooley and Hansen (1989) using a version of the representative agent cash-in-advance model, where agents have a transactions demand for money. This estimate is considerably smaller than those found here and in Imrohoroglu (1992), where agents have an asset demand for money. On the other hand, Cooley and Hansen report that the cost of a 400% inflation is over 7.5% of income in their model, where all purchases of consumption goods are made with money. This estimate is larger than the cost of hyperinflation found here for the turnpike economy with $N = 4$, where there is considerable scope for private credit arrangements that circumvent the need for money.

Table V reports the welfare costs of constant money growth rate policies when the risk aversion parameter is reduced to $\sigma = 1$, so that preferences are logarithmic. Since agents are less risk averse, the insurance-providing effects of inflationary policy are even less important than in the cases shown in Tables I–IV. With $N = 1$, welfare reaches a peak under a rate of money growth that is less than 1% per period. For $N \geq 2$, the negative effects of inflation dominate at all positive rates of money

TABLE V
CONSTANT MONETARY GROWTH: $\sigma = 1$

Money growth per period $z^{1/N}$	Welfare cost ω^*			
	$N = 1$	$N = 2$	$N = 3$	$N = 4$
1.000000	1.000000	1.000000	1.000000	1.000000
1.000001	0.999999	1.000000	1.000000	1.000000
1.00001	0.999990	1.000002	1.000001	1.000003
1.0001	0.999910	1.000015	1.000012	1.000029
1.001	0.999782	1.000380	1.000121	1.000292
1.01	1.0045	1.0048	1.0014	1.0032
1.02	1.0091	1.0075	1.0033	1.0070
1.04	1.0204	1.0154	1.0082	1.0094
1.06	1.0317	1.0251	1.0143	1.0094
1.08	1.0435	1.0301	1.0214	1.0094
1.10	1.0591	1.0331	1.0266	1.0094
1.12	1.0679	1.0367	1.0266	1.0094
∞	4.6304	1.1025	1.0266	1.0094

growth. Table V also reveals a possibility that does not arise in the earlier examples: when agents are not very risk averse and have access to limited markets for private securities, their asset demand for money can vanish completely. Monetary equilibria with $\sigma = 1$ do not exist for money growth rates in excess of 10% when $N = 3$ and 4% when $N = 4$; the welfare costs of these policies equal the welfare costs of abandoning money altogether.

Levine (1991) and Kehoe *et al.* (1992) find that when agents have an asset demand or money, the insurance-providing effects of inflationary policy can make a positive rate of money growth desirable. The results presented here, however, show that this striking result depends in large part upon assumptions that rule out the existence of private credit markets. They suggest that where money and private securities coexist, the dominant effect of inflation on welfare is likely to be the same one identified by models of the transactions demand for money: inflation causes agents to inefficiently economize on their cash balances and thereby inhibits welfare-improving exchange. The results also illustrate that the existence of private credit markets limits the government's overall ability to do both harm and good through inflationary policy.

6. THE INFORMATIONAL CONSTRAINTS ON INFLATIONARY POLICY

In each of the examples considered above, the government cannot observe the state of the economy and cannot distinguish between agent types; these informational restrictions give rise to the constraints (8) and (9) on the functions z and h. At an opposite extreme from this class of policies is the case where the government can observe and respond to the state of the economy and where it can effectively discriminate between agents of different types. In this case, both z and h can depend nontrivially on m and s. In addition, the government can choose its transfer $h(m, s)$ to type 0 agents independently of its transfer $z(m, s)$ to the economy as a whole. The government can then implement the economy's Pareto-optimal resource allocation under which all agents are treated alike, as the following proposition (proved in the Appendix) shows:

PROPOSITION. *A Pareto-optimal allocation is supported in equilibrium under the policy that sets* $z(m, s) = \beta^N$, $h(m, s) = (\beta^N - 1)[(1 - s)\omega_H + s\omega_L]$, and $M_0^0 = M_0^1 = 1/2$.

The proposition indicates that the Friedman rule is optimal in the stochastic turnpike model since, with $z(m, s) = \beta^N$, the money supply is steadily contracted so as to make the nominal interest rate equal to zero. As in Townsend's (1980) original turnpike model, however, implementing the optimal allocation through Friedman-style deflation requires that the monetary authority distinguish between agents of different types by assessing larger lump-sum taxes on those who receive the high endowment.

Intermediate between these first two cases is one in which the government can respond to the state of the economy but cannot distinguish between agent types. In this case, the function z can depend nontrivially on m and s, although maintaining the symmetry assumption requires that $z(m, s)$ satisfy

$$z(m, s) = z(1 - m, 1 - s) \tag{10}$$

for all $m \in [0, 1]$ and $s \in \{0, 1\}$. Since $H^0(s^t)$ and $H^1(s^t)$ must be identical, Eq. (7) implies that the function h must satisfy

$$h(m, s) = [z(m, s) - 1]/2 \tag{11}$$

for all $m \in [0, 1]$ and $s \in \{0, 1\}$. Equations (10) and (11) indicate that the government can no longer choose the functions $z(m, 0)$, $z(m, 1)$, $h(m, 0)$, and $h(m, 1)$ independently when it cannot distinguish between agent types. In particular, the Friedman-style policy called for by the proposition is no longer feasible.

Nevertheless, there exist policies that satisfy the informational constraints (10) and (11) that call for zero inflation on average and improve on all of the policies considered in Tables I–IV. For example, suppose that

$$z(m, 1) = [1.02 - 0.04m]^N,$$

and let $z(m, 0)$ and $h(m, s)$ be determined by (10) and (11). This policy calls for 2% money growth per period during trading sessions when agents who receive the low endowment at the beginning of the session have run their money holdings down to zero. It offsets this expansion by contracting the money supply by 2% per period when agents who receive the low endowment at the beginning of the period hold the entire money supply. Money growth lies somewhere between −2% and 2% per period when both representative agents hold a share of the money stock. Thus, the policy takes advantage of the positive redistributive effects of inflationary policy when these effects are most valuable but still maintains zero money growth on average. It may be characterized as being countercyclical, since money growth varies inversely with the volume of trade.

When the risk aversion parameter is reset to $\sigma = 2$, Table VI shows that in the turnpike economy with $N = 1$, agents prefer the countercyclical policy to even the welfare-maximizing constant money growth rate of 3% per period. The countercyclical policy is better than all of the constant growth rate policies in the economies with $N \geq 2$ as well.

These results illustrate how the welfare consequences of inflationary policy depend crucially on assumptions that limit the government's information set. When the government has little information about the economy, as in Levine (1991) and Kehoe *et al.* (1992), positive rates of inflation can be welfare-improving. When the government has more

TABLE VI
COUNTERCYCLICAL POLICY: $\sigma = 2$

Money growth per period $[z(m, 1)]^{1/N}$	Variance of consumption $V[c(m, s)]$	Average velocity $E[v(m, s)]$	Expected utility	Welfare cost ω^*
		$N = 1$		
1.00	0.0142	0.2578	−23.1909	1.0000
1.02–0.04m	0.0140	0.2983	−22.4460	0.9823
1.03	0.0163	0.3820	−22.9685	0.9947
		$N = 2$		
1.00	0.0122	0.6742	−21.2195	1.0000
1.02–0.04m	0.0110	0.7036	−20.9426	0.9931
1.0001	0.0122	0.6751	−21.2166	0.9999
		$N = 3$		
1.00	0.0104	1.3100	−20.7411	1.0000
1.02–0.04m	0.0098	1.3643	−20.6213	0.9970
1.01	0.0112	1.5619	−20.8617	1.0030
		$N = 4$		
1.00	0.0089	2.3423	−20.4401	1.0000
1.02–0.04m	0.0084	2.3967	−20.3524	0.9978
1.01	0.0100	3.1670	−20.6124	1.0044

information, however, the policy implications of the stochastic turnpike model more closely resemble those of models of the transactions demand for money; when the government can at least respond to the state of the economy, zero average inflation is better than consistently positive inflation, and when the government can distinguish between agent types as well, the Friedman rule is optimal. These results also support Sargent's (1987, p. 270) more general conclusion that one cannot assess the welfare consequences of inflation without being specific about how the government brings about the underlying monetary expansion.

7. SUMMARY AND EXTENSIONS

Levine (1991) and Kehoe *et al.* (1992) present a model of the asset demand for money in which money is the only asset and government policy is constrained to involve a constant growth rate of the money supply effected through equal lump-sum transfers to all agents. They demonstrate that in this model, inflationary policy may be desirable because of the insurance-providing aspects of monetary expansion. A special case of the stochastic turnpike model presented here is quite similar to this earlier model. Agents have an asset demand for money, the only financial asset, and the government provides constant nominal transfers to all agents. In fact, the numerical results demonstrate that inflationary policy is desirable in this special case of the turnpike model as well.

The numerical work also reveals, however, that inflation is no longer desirable when agents have limited access to private credit markets, so that government-issued money and privately issued securities coexist as means of exchange. In addition, when the government is able to respond to changes in the state of the aggregate economy, policies that feature zero money growth on average are preferable to ones that are purely expansionary. When assumptions that rule out the existence of private credit markets and constrain the flexibility of government policy are relaxed, therefore, this model of the asset demand for money has policy implications that more closely resemble those of representative agent models of the transactions demand for money: inflation causes agents to economize on their cash balances, inhibits trade, and therefore reduces welfare. Furthermore, the overall efficacy of inflationary policy to do both harm and good diminishes as the scope for private credit arrangements expands.

As noted by Moutot (1991), models that feature heterogeneous agents typically generate artificial time series that are considerably more variable than those generated by representative agent models. In Cooley and Hansen's (1989) cash-in-advance model, for example, both inflation and the income velocity of money are constant given constant money growth. In the turnpike model presented here, inflation and velocity are variable, even when money grows at a constant rate. If preferences were defined over leisure as well as consumption if ω_H and ω_L represented productivity shocks rather than endowment shocks, then the turnpike model would also display endogenous fluctuations in aggregate output similar to those found in Scheinkman and Weiss (1986). Whether government policy should be aimed at smoothing these real fluctuations is a question for future research.

Smith (1992) studies a stochastic model with heterogeneous agents and production and finds that government policy affects the real interest rate in ways that further complicate the welfare consequences of inflation. Whether these effects arise in a stochastic turnpike model with production is another question for future research. Adding production possibilities and a labor–leisure choice to the stochastic turnpike model would also allow the welfare cost of inflationary finance to be weighed against the welfare cost of other distortionary policies, including taxes on labor income. Imrohoroglu and Prescott (1991) and Cooley and Hansen (1991) suggest that a complete assessment of the inflation tax requires such a comparison.

Finally, the stochastic turnpike model presented here contains only two agent types; these types are distinct only because their endowments differ. Whether introducing additional forms of heterogeneity would yield further insights or merely complicate the analysis is unclear.

APPENDIX

Numerical Methods

Competitive equilibria for the stochastic turnpike economy are constructed by characterizing them as solutions to a system of functional equations and then solving this system of functional equations numerically.

Let the functions $c(m, s)$, $\mu(m, s)$, $z(m, s)$, $h(m, s)$, and $v(m, s)$ be as defined in the text. Substituting these functions into the representative type 0 agent's budget constraint (which will always hold with equality) yields

$$\frac{m_{t-1} + h(m_{t-1}, s_t)}{v(m_{t-1}, s_t)} + [(1 - s_t)\omega_H + s_t\omega_L] + \frac{\beta(1 - \beta^{N-1})}{2(1 - \beta)} = \frac{(1 - \beta^N)c(m_{t-1}, s_t)}{1 - \beta} + \frac{\mu(m_{t-1}, s_t)z(m_{t-1}, s_t)}{v(m_{t-1}, s_t)}, \quad \text{(A.1)}$$

where the result that $q_t(s^{t+j}) = (\beta/2)^j$ and Eqs. (3) and (4) have all been used. Substituting the functions into the stochastic Euler equation (5) for both $i = 0$ and $i = 1$ yields

$$\frac{u'[c(m_{t-1}, s_t)]}{v(m_{t-1}, s_t)} \geq \frac{\beta^N}{z(m_{t-1}, s_t)} E_t \left\{ \frac{u'[c(\mu(m_{t-1}, s_t), s_{t+N})]}{v(\mu(m_{t-1}, s_t), s_{t+N})} \right\} \tag{A.2}$$

and

$$\frac{u'[1 - c(m_{t-1}, s_t)]}{v(m_{t-1}, s_t)} \geq \frac{\beta^N}{z(m_{t-1}, s_t)} E_t \left\{ \frac{u'[1 - c(\mu(m_{t-1}, s_t), s_{t+N})]}{v(\mu(m_{t-1}, s_t), s_{t+N})} \right\}. \tag{A.3}$$

Since (A.1)–(A.3) must hold for all $m_{t-1} \in [0, 1]$ and $s_t \in \{0, 1\}$, they may be rewritten as the functional equations

$$\frac{u'[c(m, s)]}{v(m, s)} - \gamma(m, s) = \frac{\beta^N}{2z(m, s)} \left[\frac{u'[c(\mu(m, s), 1)]}{v(\mu(m, s), 1)} + \frac{u'[c(\mu(m, s), 0)]}{v(\mu(m, s), 0)} \right], \tag{A.4}$$

$$\frac{u'[1 - c(m, s)]}{v(m, s)} - \varphi(m, s) = \frac{\beta^N}{2z(m, s)} \left[\frac{u'[1 - c(\mu(m, s), 1)]}{v(\mu(m, s), 1)} + \frac{u'[1 - c(\mu(m, s), 0)]}{v(\mu(m, s), 0)} \right], \tag{A.5}$$

$$c(m, s) = \left(\frac{1 - \beta}{1 - \beta^N} \right) \left[\frac{m + h(m, s) - \mu(m, s) z(m, s)}{v(m, s)} + [(1 - s)\omega_{\mathrm{H}} + s\omega_{\mathrm{L}}] + \frac{\beta(1 - \beta^{N-1})}{2(1 - \beta)} \right], \tag{A.6}$$

$$\gamma(m, s)\mu(m, s) = 0, \qquad \gamma(m, s) \geq 0, \tag{A.7}$$

and

$$\varphi(m, s)[1 - \mu(m, s)] = 0, \qquad \varphi(m, s) \geq 0. \tag{A.8}$$

In addition, symmetry requires that

$$c(m, s) = 1 - c(1 - m, 1 - s), \tag{A.9}$$

$$\mu(m, s) = 1 - \mu(1 - m, 1 - s), \tag{A.10}$$

$$v(m, s) = v(1 - m, 1 - s), \tag{A.11}$$

and

$$\gamma(m, s) = \varphi(1 - m, 1 - s). \tag{A.12}$$

Competitive equilibria for the stochastic turnpike model are identified as solutions to the system of functional equations (A.4)–(A.12).

The solution to (A.4)–(A.12) is found numerically by approximating $\mu(m, 0)$ and $v(m, 0)$ by the piecewise cubic functions

$$\hat{\mu}(m, 0) = a_k + b_k m + c_k m^2 + d_k m^3 \tag{A.13}$$

and

$$\hat{v}(m, 0) = e_k + f_k m + g_k m^2 + h_k m^3, \tag{A.14}$$

where $k = 1$ if $m \in [0, 0.2)$, $k = 2$ if $m \in [0.2, 0.4)$, $k = 3$ if $m \in [0.4, 0.6)$, $k = 4$ if $m \in [0.6, 0.8)$, and $k = 5$ if $m \in [0.8, 1.0]$. Given the approximations $\hat{\mu}(m, 0)$ and $\hat{v}(m, 0)$, approximations $\hat{c}(m, 0)$, $\hat{\mu}(m, 1)$, $\hat{v}(m, 1)$, and $\hat{c}(m, 1)$ are constructed so that (A.6) and (A.9)–(A.11) are satisfied.

The coefficients $\{a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k\}_{k=1}^{5}$ are determined by requiring that Eqs. (A.4) and (A.5) hold at 20 values for m on $[0, 1]$; the 40 nonlinear equations are solved for the 40 unknowns. The approximations $\hat{\gamma}(m, 0)$ and $\hat{\varphi}(m, 0)$ are constructed so that (A.7) and (A.8) are satisfied at the same 20 values for m. The approximations $\hat{\gamma}(m, 1)$ and $\hat{\varphi}(m, 1)$ are constructed to satisfy Eq. (A.12). Throughout, the constraints $\hat{c}(m, s) > 0$, $1 \geq \hat{\mu}(m, s) \geq 0$, and $\hat{v}(m, s) > 0$ for all $m \in [0, 1]$ and $s \in \{0, 1\}$ are imposed. Finally, the limiting distribution λ is found by confining the approximation for $\mu(m, s)$ to the finite set $\mathbf{M} = \{0, 0.01, 0.02, \ldots 1\}$ and solving the equations

$$\lambda(m') = \frac{1}{2} \sum_{m \in \mathbf{M}_0(m')} \lambda(m) + \frac{1}{2} \sum_{m \in \mathbf{M}_1(m')} \lambda(m)$$

for all $\lambda(m')$ such that $m' \in \mathbf{M}$, where

$$\mathbf{M}_0(m') = \{m \in M: \mu(m, 0) = m'\} \quad \text{and} \quad \mathbf{M}_1(m') = \{m \in M: \mu(m, 1) = m'\}.$$

This solution procedure specializes the projection methods outlined by Judd (1992) by choosing the set of piecewise cubic functions in Eqs. (A.13)–(A.14) as a basis and a collocation method for the projection conditions. Piecewise cubic approximation allows for considerable nonlinearity in the functions c, m, and v and, in particular, for kinks in these functions associated with binding nonnegativity constraints on money holdings.

Proof of Proposition

With $z(m, s) = \beta^N$, $h(m, s) = (\beta^N - 1)[(1 - s)\omega_H + s\omega_L]$, and $M_0^0 = M_0^1 = \frac{1}{2}$,

$$M^s(s^{t+N-1}) = \beta^{t+N-1}$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $s^{t+N-1} \in \mathbf{h}^{t+N-1}$ and

$$H^i(s^t) = \beta^{t-1}(\beta^N - 1)y^i(s^t)$$

for all $t \in \mathbf{T}$ and $s^t \in \mathbf{h}^t$.

Suppose that

$$q_t(s^{t+j}) + (\beta/2)^j$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N - 1$, and $s^{t+j} \in \mathbf{h}^j(s^t)$,

$$p_t(s^t) = 1/[\beta^{t-1}(1 - \beta^N)]$$

for all $t \in \mathbf{T}$ and $s^t \in \mathbf{h}^t$, and

$$p_t(s^{t+N-1}) = 1/[\beta^{t-1}(1 - \beta^N)2^{N-1}]$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$. Then for $i \in \{0, 1\}$, the representative type i agent's problem is solved with

$$c^i(s^{t+j}) = \tfrac{1}{2}$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, $j = 0, 1, \ldots, N - 1$, and $s^{t+j} \in \mathbf{h}^j(s^t)$ and

$$m^i(s^{t+N-1}) = \tfrac{1}{2}$$

for all $t \in \mathbf{T}$, $s^t \in \mathbf{h}^t$, and $s^{t+N-1} \in \mathbf{h}^{N-1}(s^t)$. These quantities also satisfy the market clearing conditions required by the definition of competitive equilibrium.

Thus, the policy described by the proposition gives rise to an equilibrium in which each representative agent consumes half of the aggregate endowment for sure in every period. Since the Pareto-optimal allocation that treats all agents alike also supplies each representative agent with half of the aggregate endowment for sure in every period, this policy implements a Pareto optimal allocation.

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